A Change–Resistance Account of Children's Difficulties Understanding Mathematical Equivalence

Nicole M. McNeil

University of Notre Dame

ABSTRACT—***Most elementary school children in the United States have difficulties understanding mathematical equivalence in symbolic form (e.g., 3 + 4 = 5 + 2, 7 = 7). This is troubling because a formal understanding of mathematical equivalence is necessary for success in algebra and all higher level mathematics. Historically, children's difficulties with mathematical equivalence have been attributed to something that children lack relative to adults (e.g., domain-general logical structures, working memory capacity, proficiency with basic arithmetic facts). However, a change–resistance account suggests that children's difficulties are due to inappropriate generalization of knowledge constructed from overly narrow experience with arithmetic. This account has not only enhanced our understanding of the nature of children's difficulties with mathematical equivalence but also helped us identify some of the malleable factors that can be changed to improve children's understanding of this concept.***

KEYWORDS—*mathematics learning; conceptual development; arithmetic practice*

Nicole M. McNeil, University of Notre Dame, Notre Dame, IN.

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Correspondence concerning this article should be addressed to Nicole M. McNeil, Department of Psychology, University of Notre Dame, 118 Haggar Hall, Notre Dame, IN 46556; e-mail: nmcneil@nd.edu.

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Children often exhibit misconceptions when solving mathematics problems. For example, preschoolers think the volume of liquid in a beaker changes after it is poured into a taller, thinner beaker (Piaget & Szeminska, 1941/1995), and elementary school children assume that subtraction always entails subtracting the smaller digit from the larger one (Brown & VanLehn, 1980/1988). Misconceptions like these not only offer a window into how the mind works, but also inspire innovative interventions.

For decades, scientists have studied elementary school children's misconceptions in their understanding of mathematical equivalence. Mathematical equivalence is the relation between two quantities that are the same (Kieran, 1981), and its symbolic form specifies that the two sides of an equation are equal and interchangeable (e.g., 3 + 4 = 5 + 2). Formally understanding mathematical equivalence means understanding the equal sign as a relational symbol, comprising both sameness and substitutive components (Jones, Inglis, Gilmore, & Evans, 2013; Knuth, Stephens, McNeil, & Alibali, 2006; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). Children who understand mathematical equivalence do not view an arithmetic problem simply as a signal to carry out a procedure; instead, they look at the problem in its entirety and identify the relation being expressed before beginning to calculate (Jacobs, Franke, Carpenter, Levi, & Battey, 2007).

Mathematical equivalence is regarded widely as necessary for success in algebra (Carpenter, Franke, & Levi, 2003; Knuth et al., 2006; National Council of Teachers of Mathematics, 2000), but many children have trouble understanding it (Baroody & Ginsburg, 1983; Falkner, Levi, & Carpenter, 1999; Kieran, 1981). Although these difficulties are common in several countries (DeCorte & Verschaffel, 1981; Humberstone & Reeve, 2008; Jones et al., 2013; Molina, Castro, & Castro, 2009; Sherman & Bisanz, 2009), this review focuses on research conducted in the United States. Misconceptions are most apparent when children are asked to solve equations with operations on both sides of the equal sign (e.g., 3 + 7 + 5 = 3 + __; Perry, Church, & Goldin-Meadow, 1988). Although these mathematical equivalence problems are not typically included in traditional

K–8 curricula in the United States (McNeil et al., 2006; Powell, 2012; Seo & Ginsburg, 2003), adults are usually surprised to discover that only about 20% of children (aged 7–11) in this country solve the problems correctly (in comparison, more than 90% of children in China solve such problems correctly; Li, Ding, Capraro, & Capraro, 2008).

Moreover, interventions may not easily correct children's misconceptions. Some children fail to learn from interventions (e.g., Jacobs et al., 2007; Rittle-Johnson & Alibali, 1999). Others seem to learn, but then fail to transfer their knowledge to problems that differ on surface features (e.g., Alibali, Phillips, & Fischer, 2009; Perry, 1991). Still other children seem to learn and transfer, but then revert to their original incorrect ways of thinking a few weeks after learning (e.g., Cook, Mitchell, & Goldin-Meadow, 2008; McNeil & Alibali, 2000).

EXPLAINING CHILDREN'S DIFFICULTIES

Why do children have such difficulties with mathematical equivalence? Historically, researchers have attributed children's difficulties in this area to structures or functions that children lack relative to adults, such as underdeveloped domain-general logical structures for coordinating equivalence relations (Kieran, 1981; Piaget & Szeminska, 1941/1995), an immature working memory system (Case, 1978), or a lack of proficiency with basic arithmetic facts (Kaye, 1986).

Although such factors may play a role in children's difficulties with math equivalence, children's early experiences with arithmetic also play a part (Baroody & Ginsburg, 1983; Cobb, 1987; Li et al., 2008; McNeil & Alibali, 2005b; Seo & Ginsburg, 2003). Davydov (1969/1991) was the first to show that children as young as first grade could learn algebraic concepts, including mathematical equivalence. Since then, studies have shown that children in China, Korea, and Turkey better understand math equivalence than their peers of the same age in the United States (Capraro et al., 2010). Moreover, even in studies within the United States, extensive conceptual instruction can improve understanding of mathematical equivalence in some children (e.g., Baroody & Ginsburg, 1983; Jacobs et al., 2007; Saenz-Ludlow & Walgamuth, 1998). These findings suggest that under some circumstances, young children can understand mathematical equivalence.

A change–resistance account has explained how the early learning environment can hinder children's understanding of mathematical equivalence (McNeil & Alibali, 2005b). This account was inspired by classic top-down approaches to learning and cognition (e.g., Luchins, 1942; Rumelhart, 1980), and by developmental theories that emphasize the role of statistical learning in development (e.g., Rogers, Rakison, & McClelland, 2004; Saffran, 2003). According to this account, children detect and extract (often subconsciously and incidentally) the patterns they encounter routinely in traditional arithmetic and construct long-term memory representations that serve as their default representations in mathematics. Although such representations are typically beneficial, they can become entrenched, and learning difficulties arise when information to be learned overlaps with, but does not map directly onto, entrenched patterns (e.g., Bruner, 1957; Zevin & Seidenberg, 2002). This account suggests that the knowledge children construct early in a domain plays a central role in shaping and constraining development (cf. Munakata, 1998; Thelen & Smith, 1994). It attributes children's difficulties with mathematical equivalence primarily to constraints and misconceptions that emerge as a consequence of prior learning, rather than to general conceptual, procedural, or working memory limitations in childhood.

Consistent with the change–resistance account, studies have shown that children's difficulties with mathematical equivalence stem from their reliance on patterns encountered routinely in arithmetic (McNeil & Alibali, 2004, 2005b). In the United States, children learn arithmetic in a procedural fashion for years before they learn to reason relationally about equations. Moreover, arithmetic problems are usually presented with operations to the left of the equal sign and the answer to the right (e.g., $3 + 4 = 7$; McNeil et al., 2006; Seo & Ginsburg, 2003), a format that fails to highlight the interchangeable nature of the two sides of an equation. As a result of this narrow experience, children extract at least three operational patterns that do not generalize beyond arithmetic (McNeil & Alibali, 2005b): First, children learn to expect all operations to be on the left side of a math problem followed by the equal sign and an answer blank at the end (Alibali et al., 2009; Cobb, 1987; McNeil & Alibali, 2004). Second, they learn to perform all given operations on all given numbers (McNeil & Alibali, 2005b). And third, they learn to interpret the equal sign operationally as a symbol to do something (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Kieran, 1981; McNeil & Alibali, 2005a). As these representations become entrenched, children rely on them as their default representations when they encounter novel mathematics problems.

Although relying on these operational patterns may be helpful when children are given traditional arithmetic problems (e.g., $3 + 4 = \_\_$), it is detrimental when children have to encode, interpret, or solve problems of mathematical equivalence. For example, when asked to reconstruct the problem $7 + 4 + 5 = 7 + \_\_$ after viewing it briefly, many children claim that they saw operations on the left side and write $7 + 4 + 5 + 7 = \_\_$ (McNeil & Alibali, 2004). When asked to define the equal sign—even in the context of a mathematical equivalence problem—many children treat it like an arithmetic operator (like + or −) that means they should calculate the total (McNeil & Alibali, 2005a). When asked to solve the problem $7 + 4 + 5 = 7 + \_\_$, many children perform all given operations on all given numbers and put 23 (instead of 9) in the blank (McNeil, 2007; Rittle-Johnson, 2006). These findings support the change–resistance account and suggest that children's difficulties with mathematical equivalence are due partly to inappropriate generalization of knowledge constructed from overly narrow experience with traditional arithmetic.

NOVEL PREDICTIONS OF THE CHANGE–RESISTANCE ACCOUNT

The change–resistance account also allows us to make novel predictions, many of which have been supported empirically. For example, most theories predict that performance on math equivalence problems should improve with age. Indeed, "'performance improves with age' is as close to a law as any generalization that has emerged from the study of cognitive development" (Siegler, 2004, p. 2). However, the change–resistance account predicts—and studies have shown—that performance actually declines in the early school years before it improves (McNeil, 2007). This is because, as children progress from first to third grade, they continue to gain narrow practice with arithmetic, so they are strengthening the very knowledge structures hypothesized to hinder understanding of mathematical equivalence. Thus, similar to other U-shaped developmental patterns—such as scale errors (DeLoache, Uttal, & Rosengren, 2004), the ability to map arbitrary gestures to referents (Namy, Campbell, & Tomasello, 2004), and the ability to process contradictory messages simultaneously (Church, Kelly, & Lynch, 2000)—understanding mathematical equivalence does not follow the tenet that performance improves with age.

The change–resistance account also challenges the widespread belief that practice with basic arithmetic facts improves performance on higher level math problems. This belief is rooted in the decomposition thesis (Anderson, 2002), which suggests that a complex skill can be decomposed into component subskills and that practice on those subskills facilitates learning and carrying out the complex skill. The logic is simple: When learners lack proficiency with the subskills, their cognitive resources are committed to the step-by-step execution of those subskills and are largely unavailable for other processes, such as encoding novel problem formats or generating new strategies. In contrast, when learners have sufficient practice with subskills, cognitive resources can be allocated to other processes (e.g., Kaye, 1986). These ideas have been invoked to advocate for back-to-basics math instruction, which maintains that performance in algebra can be improved by drilling children on arithmetic facts until they are proficient. However, the change–resistance account predicts that concentrated practice with traditional arithmetic hinders understanding of mathematical equivalence because it activates and strengthens narrow representations of the operational patterns.

A series of experiments with undergraduates who had attended elementary school in the United States (McNeil, Rittle-Johnson, Hattikudur, & Petersen, 2010) supported this prediction. Participants were randomly assigned either to an arithmetic practice condition (e.g., 3 + 4) or to one of several control conditions (e.g., no input, color mixing, algebra practice). Then they solved mathematical equivalence problems under speeded conditions. As predicted, participants were less likely to solve a mathematical equivalence problem correctly after practicing arithmetic than after participating in one of the control conditions. This result suggests that practice with arithmetic activates overly narrow representations that hinder performance on mathematical equivalence problems. It also suggests that even educated adults, who have years of experience with arithmetic and algebra, have not fully integrated their knowledge of arithmetic with their knowledge of algebra.

The consequences of traditional arithmetic practice are unacceptable, but eliminating arithmetic practice altogether is not a viable alternative. Children need to know arithmetic before they can solve higher order mathematics problems correctly. However, acquiring operational patterns is not inevitable. As mentioned previously, children in China do not extract operational patterns (Li et al., 2008), and undergraduates who attended elementary school in Asian countries do not solve mathematical equivalence problems incorrectly under speeded conditions, even after practicing arithmetic (McNeil et al., 2010).

The change–resistance account predicts—and research has shown—that understanding mathematical equivalence can be improved by modifying arithmetic practice to be less narrow and more in line with the underlying concepts. Specifically, three modifications help: (a) presenting arithmetic problems in a nontraditional format that puts operations on the right side (e.g., __ = 9 + 8; McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011), (b) organizing problems into practice sets based on equivalent values (e.g., 2 + 5 = __, 3 + 4 = __, 6 + 1 = __; McNeil et al., 2012), and (c) using relational terms such as *is equal to* and *is the same amount as* in place of the equal sign in some practice problems (Chesney, McNeil, Petersen, & Dunwiddie, 2014).

These three modifications were incorporated into a nontraditional arithmetic practice workbook and compared experimentally to an analogous traditional arithmetic practice workbook in second-grade classrooms (McNeil, Fyfe, & Dunwiddie, 2014). Children within classrooms were randomly assigned to use one of the two workbooks for 15 min a day, 2 days a week, for 12 weeks. As predicted, children who used the nontraditional workbook had a greater understanding of mathematical equivalence than children who used the traditional workbook, and this advantage persisted for about 5–6 months after the workbook practice ended. Thus, simple changes to the format and organization of arithmetic practice in a naturalistic classroom can improve children's understanding of mathematical equivalence.

Although modified arithmetic practice helps children understand mathematical equivalence, such modifications may be inadequate to eradicate reliance on entrenched operational patterns. Children in the United States may interpret addition informally as a unidirectional process even before they start formal schooling (Baroody & Ginsburg, 1983), and they start to apply the operational patterns to arithmetic problems as early as first grade (e.g., Falkner et al., 1999). Consequently, in these children, arithmetic problems may activate representations of the operational patterns to some degree, regardless of the format in which the problems are presented. Thus, when teaching children about the equal sign, teachers may need to remove arithmetic altogether and present the equal sign in other contexts (e.g., 28 = 28) first, so children can solidify a relational view before moving on to a variety of formats of arithmetic problems (Baroody & Ginsburg, 1983; Renwick, 1932).

This is how the equal sign is introduced in China where, as stated, more than 90% of children solve mathematical equivalence problems correctly (Capraro et al., 2010). This discrepancy in understanding is partly due to differences in both the format and sequence of problems that children learn (Li et al., 2008). For example, in contrast to mathematics textbooks in the United States, math textbooks in China often introduce the equal sign in a context of equivalence relations first and only later embed the sign within mathematical equations involving arithmetic operators and numbers. A classroom-based experiment in the United States supported this idea (McNeil, 2008). Children were randomly assigned to learn about the meaning of the equal sign while looking at either arithmetic problems (e.g., 15 + 13 = 28) or nonarithmetic problems (e.g., 28 = 28). As predicted, children learned more when lessons were given outside of an arithmetic context than when they were given in an arithmetic context. Thus, educators may want to introduce the equal sign in the context of equivalence relations before embedding it within equations involving arithmetic.

LOOKING AHEAD

Despite progress over the past two decades in understanding children's difficulties with mathematical equivalence, at least three critical questions remain. First, what are the origins of individual differences in children's early understanding of mathematical equivalence? We know that most children in the United States struggle to understand mathematical equivalence; however, a substantial minority develops an accurate formal understanding, despite attending the same schools and having the same narrow experiences with arithmetic. Researchers have not addressed the factors that give rise to these individual differences. Longitudinal designs should assess which skills at the start of formal schooling predict children's understanding of mathematical equivalence in elementary school.

Second, what are the long-term consequences of having difficulties with mathematical equivalence? Most researchers assume that a greater understanding of mathematical equivalence in the early grades leads to more success in mathematics as children progress through school, into algebra, and beyond. However, this assumption has never been directly tested, making it difficult to determine if improving children's understanding should be a priority. Researchers should assess whether children's understanding of math equivalence in the early school years predicts math achievement and algebra readiness in subsequent years, after controlling for other predictors such as IQ and socioeconomic status.

Third, what combination of lessons and activities helps all children achieve deep, long-lasting understanding of mathematical equivalence? Several small-scale component interventions have helped improve some children's understanding of mathematical equivalence when compared to control interventions, but none has produced mastery-level understanding in most children. Researchers must work with teachers to develop a comprehensive intervention that helps all children construct a mastery–level understanding of math equivalence.

CONCLUSION

Many children struggle to understand mathematical equivalence. Children's misconceptions were once attributed to something that children lack relative to adults. However, the change–resistance account attributes them, in part, to the inappropriate generalization of overly narrow arithmetic knowledge. This account not only highlights the role of the early learning environment in establishing misconceptions that shape and constrain development, but also refines our understanding of the basic psychological processes involved in mathematical thinking (e.g., how practice with subskills affects performance, how new knowledge is integrated with old knowledge). The account also makes unique predictions about ways to structure the learning environment to help children learn mathematical equivalence, thus providing research-based solutions to a critical educational problem.

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