RUNNING HEAD: NONTRADITIONAL ARITHMETIC PRACTICE

Benefits of practicing 4 = 2 + 2: Nontraditional problem formats facilitate children's understanding of mathematical equivalence

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Final version accepted at *Child Development*: May 19, 2010

Cut to 40 pages: April 8, 2011

Word count: 8156

Abstract

This study examined if practice with arithmetic problems presented in a nontraditional problem format improves understanding of mathematical equivalence. Children (*M* age = 8;0; *N* = 90) were randomly assigned to practice addition in one of three conditions: (a) traditional, in which problems were presented in the traditional "operations on left side" format (e.g., 9 + 8 = 17), (b) nontraditional, in which problems were presented in a nontraditional format (e.g., 17 = 9 + 8), or (c) no extra practice. Children developed a better understanding of mathematical equivalence after receiving nontraditional practice than after receiving traditional practice or no extra practice. Results suggest that minor differences in early input can yield substantial differences in children's understanding of fundamental concepts.

Benefits of practicing 4 = 2 + 2: Nontraditional problem formats facilitate children's understanding of mathematical equivalence

Decades of research in cognitive development and mathematics education have shown that children struggle to understand mathematical equivalence, particularly in symbolic form (e.g., Alibali, 1999; Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; McNeil, 2008; Renwick, 1932). *Mathematical equivalence* is the relation between two quantities that are interchangeable (Kieran, 1981), and its symbolic form specifies that the two sides of a mathematical equation are equal and interchangeable. Mathematical equivalence is arguably one of the most important concepts for developing young children's algebraic thinking (Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali, 2006).

Difficulties with mathematical equivalence are most apparent when children are asked to solve equations that have operations on both sides of the equal sign (e.g., 3 + 7 + 5 = 3 + __), henceforth referred to as "mathematical equivalence problems." Although mathematical equivalence problems are not typically included in traditional K-8 curricula (McNeil et al., 2006; Seo & Ginsburg, 2003), most people are shocked to discover that children (ages 7-11) solve the problems incorrectly. Across nine studies, McNeil (2005) found that the vast majority of children (about 82%) did not succeed on the problems. Children's difficulties with mathematical equivalence have been shown to be robust and long-term, persisting among some middle school, high school, and even college students (Knuth et al., 2006; McNeil & Alibali, 2005a, Renwick, 1932). This is cause for concern because individuals who do not develop a correct understanding of mathematical equivalence will have difficulties advancing in mathematics and science.

Although there is growing evidence that children have difficulties with mathematical equivalence, the mechanisms underlying these difficulties and the eventual emergence of correct understanding are less clear. A central goal of research in cognitive development is to characterize how knowledge is constructed over time. To achieve this goal, we need to move beyond simply assessing children's successes and failures on mathematical tasks toward providing detailed accounts of *why* mathematical equivalence is particularly difficult to learn and *how* such difficulties are ultimately overcome (cf. Siegler, 2000).

Historically, children's difficulties with mathematical equivalence have been attributed to domain general conceptual limitations in childhood (Collins, 1974, as cited in Kieran, 1980; Piaget & Szeminska, 1995/1941), or to developmental changes in some parameter of the working memory system such as total capacity, efficiency, or processing speed (e.g., Case, 1978). However, several researchers now posit that difficulties are due, at least in part, to children's early experiences with mathematics (e.g., Baroody & Ginsburg, 1983; McNeil & Alibali, 2005b; Seo & Ginsburg, 2003; Sherman & Bisanz, 2009). Some of the first evidence came from Davydov and colleagues (Davydov, 1991/1969); they showed that first and second grade children who participated in an experimental early algebra curriculum could learn to understand algebraic concepts, including mathematical equivalence. Since then, international studies have shown that children in China, Korea, Turkey, and the Canadian province of Quebec perform better than same-aged children in the United States on mathematical equivalence problems (Capraro et al., 2010; Freiman & Lee, 2004). Even studies within the U.S. have shown that children's understanding of mathematical equivalence can be improved after several months of targeted conceptual instruction (e.g., Baroody & Ginsburg, 1983; Carpenter, Levi, & Farnsworth, 2000; Jacobs et al., 2007; Saenz-Ludlow & Walgamuth, 1998). Taken together, this work suggests that children's early learning environments may play a large role in the development of children's understandings and misunderstandings of mathematical equivalence.

McNeil and Alibali (2005b) recently extended this work by developing a theory of how the early learning environment affects the development of children's understanding of mathematical equivalence. They proposed a change-resistance account of children's difficulties with mathematical equivalence. The general idea is that learners (often subconsciously and incidentally) detect and extract the patterns routinely encountered in a domain and construct long-term memory representations that serve as the default representations in that domain. These default representations avoid unnecessary computations in the future (cf. Salthouse, 1991). While such representations are typically beneficial (e.g., Chase & Simon, 1973), they can become entrenched, and learning difficulties arise when to-be-learned information overlaps with, but does not map directly onto, entrenched patterns (e.g., Bruner, 1957; Zevin & Seidenberg, 2002).

McNeil and Alibali's (2005b) change-resistance account is consistent with other theories that focus on mechanisms of change resistance in development. Examples include Munakata's (1998) strong latent representations, dynamic systems theory's deep attractor states (Thelen & Smith, 1994), and the phenomenon of entrenchment in a connectionist model (Zevin & Seidenberg, 2002). On the whole, such theories emphasize the importance of children's early experience and practice in a domain, and they argue that the knowledge children construct early on plays a central role in shaping and constraining the path of development. In short, these theories attribute children's learning difficulties to constraints that emerge as a consequence of prior learning, rather than to general conceptual or working memory limitations in childhood.

Although a change-resistance account suggests that constraints emerge as a consequence of general learning processes that can affect learning at any age, it emphasizes the power of learners' initial associations in a domain because such associations have the potential to shape all subsequent learning. Consequently, it suggests that we should monitor the constraints that are emerging in academic domains during children's first few years of formal schooling to make sure they map onto long-term learning goals. One implication of this theoretical perspective is that children may not need months of explicit conceptual instruction to understand a difficult concept, as long as the patterns they learn early on in a domain facilitate (rather than hinder) acquisition of the to-be-learned concept.

Applied to the domain of mathematics, a change-resistance account suggests that difficulties with mathematical equivalence stem not from general conceptual or working memory limitations in childhood, but from children's representations of patterns routinely encountered in the first few years of formal arithmetic instruction (Baroody & Ginsburg, 1983; McNeil & Alibali, 2005b; Seo & Ginsburg, 2003). In the U.S., children learn arithmetic in a procedural fashion for years before they learn to reason about equations relationally, as expressions of mathematical equivalence. Moreover, arithmetic problems are almost always presented with operations to the left of the equal sign and the "answer" to the right (e.g., 3 + 4 = 7, McNeil et al., 2006; Seo & Ginsburg, 2003). This format does not highlight the interchangeable nature of the two sides of an equation (Weaver, 1973). As a result of this overly narrow experience, children extract at least three patterns that do not generalize beyond arithmetic. Such patterns have been called *operational patterns* in past work (e.g., McNeil & Alibali, 2005b) because they are derived from experience with arithmetic operations, and they reflect operational rather than relational thinking (cf. Jacobs et al., 2007). First, children learn a perceptual pattern related to the format of mathematics problems, namely the "operations on left side" format (Alibali, Phillips, & Fischer, 2009; Cobb, 1987; McNeil & Alibali, 2004). Second, children learn the problem-solving strategy ‘‘perform all given operations on all given numbers’’ (McNeil & Alibali, 2005b; Perry et al., 1988). Third, children learn to interpret the equal sign operationally as a “do something” symbol (Baroody & Ginsburg, 1983; Behr et al., 1980; Kieran, 1981; McNeil & Alibali, 2005a). Children’s representations of these operational patterns gain strength during the first few years of formal schooling and strengthen to their most entrenched levels around age nine (McNeil, 2007). Children come to rely on these operational patterns as their default representations when solving mathematics problems.

Although it may be helpful for children to rely on the operational patterns when working on traditional arithmetic problems (e.g., 3 + 4 = __), it is unhelpful when they have to encode, interpret, or solve mathematical equivalence problems. For example, when asked to reconstruct the problem “7 + 4 + 5 = 7 + __” after viewing it briefly, many children rely on their knowledge of the “operations on left side” problem format and write “7 + 4 + 5 + 7 = __” (Alibali et al., 2009; McNeil & Alibali, 2004). When asked to define the equal sign in a mathematical equivalence problem, many children say that it is an arithmetic operator (like + or -) that means “calculate the total” (McNeil & Alibali, 2005a). When asked to solve the problem “7 + 4 + 5 = 7 + __”, many children rely on their knowledge of the “perform all given operations on all given numbers” strategy and put 23 (instead of 9) in the blank (McNeil, 2007; McNeil & Alibali, 2005b; Falkner et al., 1999; Perry, 1991; Rittle-Johnson, 2006). Additionally, the strength of children’s reliance on these three operational patterns influences whether or not children will benefit from instruction on mathematical equivalence (McNeil & Alibali, 2005b). Thus, from the perspective of a change-resistance account, we can help children develop a better understanding of mathematical equivalence simply by reducing their exposure to the overly narrow forms of arithmetic practice that reinforce the operational patterns. Ideally, this type of intervention should be administered before children's representations of the operational patterns strengthen to their most entrenched levels around age nine. By re-structuring children's early experience, we can reduce the constraints that typically emerge as a consequence of learning arithmetic and, thus, make it possible for children to develop an understanding of mathematical equivalence, even without months of explicit conceptual instruction or developmental improvements in some parameter of the working memory system.

In line with this view, some educators have developed highly innovative teacher professional development programs and classroom activities designed to re-structure children's early experience with arithmetic and bring out its algebraic character (e.g., Carpenter et al., 2003; De Corte & Verschaffel, 1981; Jacobs et al., 2007; Saenz-Ludlow & Walgamuth, 1998; Schliemann et al., 2007). By and large, these programs have been successful—children who participated in the programs (or whose teachers participated in the professional development) improved their understanding of algebraic concepts, including mathematical equivalence.

Although some of these programs have been successful, they have had several limitations. For example, they have required teams of teachers and researchers who were highly trained and highly motivated. Such programs were also time intensive, requiring many hours of teacher preparation and class time. From the perspective of developmental theory, the most crucial limitation is that the programs have made it impossible to pinpoint the specific factors that caused improvements in children's understanding of mathematical equivalence. Was it the reduction in children's exposure to the overly narrow forms of arithmetic practice, or was it the explicit conceptual instruction, activities, and discussions? Because the programs were all administered as wholes, it is unclear which aspects of the programs were responsible for children's conceptual gains. It is important for theoretical reasons to determine the factors that help children develop an understanding of mathematical equivalence, so we can identify mechanisms of cognitive change and change resistance in the development of mathematical cognition. It is also important for practical reasons to know if all aspects of the programs are necessary, or if the same results could be achieved more efficiently. Bottom line: we need systematic experiments to tease apart the factors responsible for the success of such programs.

We propose that manipulating one factor—the narrowness or broadness of arithmetic practice—could address prior limitations and provide a bridge between developmental theory and educational practice. In the present experiment, we focused specifically on manipulating the narrow problem format. As mentioned above, children typically see the operations to the left of the equal sign and the "answer" to the right (e.g., 3 + 4 = 7; McNeil et al., 2006; Seo & Ginsburg, 2003). According to a change resistance account, experience with this traditional problem format reinforces children's representations of the operational patterns and constrains the development of children's understanding of mathematical equivalence. This account suggests that children's understanding of mathematical equivalence can be improved by practicing problems that are not presented in this traditional format. Thus, in the present study, we hypothesized that children would develop a better understanding of mathematical equivalence after practicing arithmetic problems presented in a nontraditional format (e.g., 17 = 9 + 8) than after practicing problems presented in the traditional format (e.g., 9 + 8 = 17).

This hypothesis not only follows directly from the predictions of a change-resistance account, but also corresponds to the recommendations of educators. Indeed, mathematics educators have long called for more diverse, richer exposure to a variety of problem types from the beginning of formal schooling (e.g., Blanton & Kaput, 2005; Hiebert et al., 1996; NCTM, 2000). Several of these experts have suggested that children may benefit from seeing nontraditional arithmetic problem formats (Baroody & Ginsburg, 1983; Denmark, Barco, & Voran, 1976; Carpenter et al., 2003; MacGregor & Stacey, 1999; Seo & Ginsburg, 2003; Weaver, 1973). However, the present study was the first well-controlled experiment to test if this relatively simple modification to arithmetic practice actually helps children develop a better understanding of mathematical equivalence. If practice with problems presented in a nontraditional format does help children develop a better understanding of mathematical equivalence, it will highlight an important bridge between developmental theory and educators' recommendations. It will also support the idea that seemingly minor differences in children's early input can play a central role in shaping and constraining the path of development of children's understanding of fundamental concepts.

Method

Participants

Participants were 100 7- and 8-year old children recruited from a diverse range of public and private elementary schools (schools ranged from 0-89% in terms of the percentage of children who qualified for free/reduced price lunch). We targeted children in this age range because they are old enough to practice arithmetic without extensive instruction, but are young enough that their representations of the operational patterns have not yet strengthened to their most entrenched levels (McNeil, 2007). Ten children were excluded from the analysis either because they did not complete one or more of the tasks ($N = 8$), or because they were so far behind grade level in mathematics that the sessions had to be altered dramatically to meet their needs ($N = 2$; one child was inconsistent in identifying Arabic numerals [1-9], and one child could not add two single-digit numbers together unless both addends were ≤ 5). Thus, the final sample consisted of 90 children (M age = 8 yrs, 0 mo; 48 boys, 42 girls; 29% African American or black, 1% Asian, 9% Hispanic or Latino, 61% white) from a midwestern U.S. city. Approximately 29% received free/reduced price lunch, and only 19% attended elementary schools in which 0% of children at the school qualified for free/reduced lunch.

Design

The design was a posttest-only randomized experiment. Children were randomly assigned to one of three conditions: (a) *traditional practice*, in which problems were presented in the traditional "operations on left side" format, such as 9 + 8 = __, (b) *nontraditional practice*, in which problems were presented in a nontraditional format, such as __ = 9 + 8, or (c) *no extra practice*, in which children did not receive any practice over and above what they ordinarily receive at school and home.

Although our use of random assignment ensured that any differences between conditions at the outset of the experiment could be attributed to chance, we nonetheless compared children in the three conditions to make sure they did not differ in terms of their background characteristics. Importantly, the conditions were well matched. We found no statistically significant differences between children in the three conditions in terms of age (in months), $F(2, 85) = 0.20$, $p = .82$; gender, $\chi^2\ (2, N = 90) = 1.88$, $p = .39$; ethnicity, $\chi^2\ (6, N = 90) = 7.35$, $p = .29$; free or reduced price lunch, $F(2, 87) = 1.11$, $p = .33$; or enrollment in a school in which 0% of children at the school qualified for free or reduced lunch, $\chi^2\ (2, N = 90) = 0.15$, $p = .93$.

All problems in the practice sessions were single-digit addition problems with two addends (i.e., $a + b = c$ or $c = a + b$). Thus, the practice sessions did not include any practice with mathematical equivalence problems prior to the assessments. After receiving practice (or no input in the "no extra practice" condition), children completed posttest measures of understanding of mathematical equivalence and computational fluency. A posttest-only design was necessary to avoid pretest sensitization to our assessment measures. Our goal was to examine if a nontraditional form of practicing addition facts causes children to construct a better understanding of mathematical equivalence. Thus, it was essential that children not be exposed to our particular measures of interest prior to receiving an intervention and posttest.

Procedure

Children in the two practice conditions participated individually in four sessions. During the first three sessions, children practiced addition facts by playing games one-on-one with a tutor (all three sessions) and by answering flashcards (first two sessions). In between these sessions, children practiced by completing brief paper-and-pencil homework assignments. Overall, children in the practice conditions received approximately 100 minutes of practice prior to completing the assessments. After the final practice session was complete, children were introduced to a new experimenter who assessed their understanding of mathematical equivalence and computational fluency (see Materials section). This new experimenter was used for the purposes of assessment, so he or she could be kept uninformed of children's practice condition. Approximately two weeks after the third session, children in the practice conditions participated in a brief follow-up session.

Practice materials

Practice sessions. The sessions were designed to help children practice solving single-digit addition facts with two addends (e.g., 9 + 8 = 17, 8 + 6 = 14). Children received practice via three main types of activities: (a) two-player games involving cards or dice, (b) flashcards, and (c) a computer game. An example of a two-player game is called "Smack it!" An illustration of the set-up for this game appears in Figure 1. The child and tutor each used a swatter with a suction cup on the end. At the beginning of the game, four addition problems were placed face-up on the table, and a pile of number cards was placed face down. To start each round, the tutor turned over one of the number cards to serve as the target number. The goal was to be the first player to “smack” the addition problem that should have the target number in the blank. Children also played other two-player games that were similar in content and scope. The specific games played were the same in both practice conditions, and the games were rigged so all children practiced the same problems. The only difference between conditions was the format in which the problems were presented (traditional versus nontraditional).

Children also practiced flashcards during the sessions. Before completing the flashcards, children received a brief demonstration on how to solve the flashcards. The tutor first presented a simple flashcard (e.g., “1 + 1 = __”). Next, the tutor read the problem aloud (e.g., “one plus one is equal to blank”). Then, the tutor provided the following instructions: “So I need to figure out what number I need to put in the blank to make this side of the equal sign (gestured to indicate left side of the problem) the same amount as (point to the equal sign) this side of the equal sign (gestured to indicate right side of the problem).” Finally, the tutor offered the correct number (e.g., “It’s two! Two should go in the blank because one plus one is equal to two.”). The demonstration was similar in the nontraditional condition, except the instructions conformed to the nontraditional problem format (e.g., “__ = 1 + 1” instead of “1 + 1 = __”). When solving flashcards, children were instructed to read each problem aloud before stating the number that should go in the blank. Children completed 26 flashcards in the first practice session and 27 flashcards in the second practice session.

Children also played a computer game in each of the three practice sessions. The game was a modified version of *Snakey Math* from Curry K. Software. The child and the tutor each controlled one of four snakes on the screen (the other two were computer controlled). Snakes could move up, down, left, and right. At the beginning of each round, a problem (e.g., 9 + 8 = __ or __ = 9 + 8) was presented at the bottom of the screen and four numbers appeared in random locations on the screen. The goal was to be the first snake to eat the number that should go in the blank.

Homework. Children were asked to continue their practice in between practice sessions through brief homework assignments. These assignments were paper-and-pencil worksheets designed to take approximately 15 minutes to complete. All problems on the worksheets were single-digit addition problems with two addends (i.e., $a + b = c$ or $c = a + b$). Thus, in line with the practice sessions, the worksheets did not include any mathematical equivalence problems. Children were told that they would receive a sticker for completing the worksheets and turning them in when they returned for the next session. When a child turned in his or her completed worksheet, the tutor asked the child to read each problem aloud (along with the number written in the blank), and the tutor corrected any errors. Errors were rare; they occurred only on 1% of all problems solved by all children across both worksheets. To ensure that all children would be able to participate in this "read aloud" activity, tutors kept correctly completed worksheets on hand. Thus, children who had not turned in their own completed worksheet participated in the "read aloud" activity by using a worksheet that had been correctly completed for them. Most children turned in their own completed worksheets (90% completion rate from the first to second session, and 68% from the second to third session), and many who did not insisted that they had done the homework, but forgot to bring it with them. There were only five children overall (3 nontraditional and 2 traditional) who failed to turn in both worksheets, and the number of children who failed to turn in both, one, or none of the worksheets did not differ across the two practice conditions, χ^2 (2, N = 60) = 1.56, p = .46.

Assessments and Coding

Measures of understanding of mathematical equivalence. Children completed three measures to assess their understanding of mathematical equivalence: (a) equation solving, (b) equation encoding, and (c) defining the equal sign. According to the nomological network defined by a change-resistance account (McNeil & Alibali, 2005b), these three measures tap a system of three distinct, but theoretically related constructs involved in children's understanding of mathematical equivalence. We established inter-rater reliability on the measures by having a second coder code the responses of 20% of the children.

To assess *equation solving*, children were videotaped as they solved and explained four mathematical equivalence problems (1 + 5 = __ + 2, 7 + 2 + 4 = __ + 4, 2 + 7 = 6 + __, 3 + 5 + 6 = 3 + __). An experimenter placed each equation on an easel and said, "Try to solve the problem as best as you can, and then write the number that goes in the blank." After children wrote a number in the blank, the experimenter said, "Can you tell me how you got *x*" (*x* denotes the given answer) (cf. Alibali, 1999; Perry, 1991; Rittle-Johnson & Alibali, 1999; Siegler, 2002). Children's problem-solving strategies were coded as correct or incorrect based on a system used in previous research (e.g., McNeil & Alibali, 2004; Perry et al., 1988). For most problems, correctness could be inferred from the solution itself (e.g., for the problem 3 + 5 + 6 = 3 + __, a solution of 17 indicated an incorrect "add all" strategy and a solution of 11 indicated a correct strategy). If the solution was ambiguous, then strategy correctness was coded based on children's verbal explanation (e.g., for the problem 3 + 5 + 6 = 3 + __, the explanation "I added 3 plus 5" indicated an incorrect strategy and the explanation "I added 5 plus 6" indicated a correct strategy). The internal consistency of scores on this measure was high, Cronbach's alpha = .91.

Inter-rater reliability was also high; agreement between coders was 99% for coding whether or not a given strategy was correct.

To assess *equation encoding*, children were asked to reconstruct four mathematical equivalence problems (7 + 1 = __ + 6, 3 + 5 + 4 = __ + 4, 4 + 5 = 3 + __, 2 + 3 + 6 = 2 + __) after viewing each for 5 seconds (cf. Chase & Simon, 1973; Siegler, 1976). Children's encoding performance was coded as correct or incorrect based on a system used in previous research (e.g., McNeil & Alibali, 2004; Rittle-Johnson & Alibali, 1999). Some common errors included converting the problem to a traditional addition problem (e.g., reconstructing 4 + 5 = 3 + __ as "4 + 5 + 3 = __"), omitting the plus on the right side of the equal sign (e.g., reconstructing 4 + 5 = 3 + __ as "4 + 5 = 3 __"), and omitting the equal sign (e.g., reconstructing 7 + 1 = __ + 6 as "7 + 1 __ + 6"). The internal consistency of scores on this measure was adequate, Cronbach's alpha = .70. Inter-rater reliability was high; agreement between coders was 100% for coding whether or not a given reconstruction was correct.

To assess *defining the equal sign*, children were videotaped as they responded to a set of questions about the equal sign. The experimenter pointed to an equal sign and asked: (1) "What is the name of this math symbol?" (2) "What does this math symbol mean?" and (3) "Can it mean anything else?" (cf. Behr et al., 1980; Baroody & Ginsburg, 1983; Knuth et al., 2006). Children's responses were categorized according to a system used in previous research (e.g., Knuth et al., 2006; McNeil & Alibali, 2005a, b). We were specifically interested in whether or not children defined the equal sign relationally as a symbol of mathematical equivalence (e.g., "two amounts are the same"). Defining the equal sign relationally has had good concurrent validity in previous research (e.g., Knuth et al., 2006). Inter-rater reliability in the present study was high; agreement between coders was 100% for coding whether or not a given definition was relational.

Two weeks after the final session, children in the practice conditions participated in a brief *follow-up assessment* of their understanding of mathematical equivalence. Children solved the same four mathematical equivalence problems that they had solved during the original assessment. However, children were provided with brief tutelage and feedback at follow-up. When the tutor presented each equation, he or she instructed the child as follows: “You need to figure out what number goes in the blank (point to blank) to make this side of the equal sign (gesture to the left-hand side of the problem) the same amount as (point to the equal sign) this side of the equal sign (gesture to the right-hand side of the problem).” If the child provided the correct number, the tutor gave positive feedback such as “good job” or “that’s right” and then moved on to the next problem. However, if the child provided an incorrect number, the tutor instructed the child as follows: “No, that’s not the number that goes in the blank. The correct number is x because a plus b is equal to x plus y” (the actual numbers in the problem are used in place of a, b, x, and y). The purpose of this follow-up session was to examine potential enduring effects of the traditional and nontraditional practice on children’s openness to learn from brief instruction on mathematical equivalence.

Measures of computational fluency. In addition to the measures of mathematical equivalence, children also completed two measures of computational fluency. The first measure was the Math Computation section of Level 8 of the Iowa Tests of Basic Skills, which is a standardized, timed measure of arithmetic computation that yields a percentile rank. Children also completed another measure of arithmetic skill (cf. Geary, Bow-Thomas, Liu, Siegler, 1996; Siegler, 1988). Students were videotaped as they solved a set of addition problems. Problems were presented one at a time in the center of a computer screen. For each trial, a fixation point appeared in the center of the screen followed by the prompts "ready," "set," and "go," displayed for 1 second each. The two addends were then presented on the screen (e.g., 9 + 8), and they remained on the screen until the student said his or her answer aloud. Reaction time was recorded. Additionally, after each trial, students were asked: "Can you tell me how you got *x*?" (*x* denotes the given answer).

Results

According to the nomological network defined by McNeil and Alibali's (2005b) change-resistance account, equation solving, equation encoding, and defining the equal sign are three distinct, but theoretically related constructs involved in children's understanding of mathematical equivalence. Thus, scores on the measures should be moderately correlated with one another. Consistent with this account, scores on the measures were correlated: equation solving and equation encoding, $r = .56$, $p < .001$; equation solving and defining the equal sign, $r = .35$, $p = .001$, equation encoding and defining the equal sign, $r = .21$, $p = .05$. This pattern of correlations is consistent with previous research (McNeil & Alibali, 2004, 2005b; Rittle-Johnson & Alibali, 1999) and provides some evidence of construct validity (Cronbach & Meehl, 1955).

Table 1 presents children's performance on each of the measures of understanding of mathematical equivalence by condition. As shown in the table, the effect of condition was the same across all three measures, so we created a composite measure for more efficient presentation. Children received one point if they scored above the average on the measure of equation solving, one point if they scored above the average on the measure of equation encoding, and one point if they provided a relational definition on the measure of equal sign understanding. Note that the results were robust and did not depend on how we assigned points for the composite measure (e.g., above the average, above the median, at least one correct, etc.). Scores on the composite measure ranged from 0 to 3 (M = 0.70, SD = 0.97).

We performed an analysis of variance (ANOVA) with condition (traditional practice, nontraditional practice, or no extra practice) as the independent variable and score on the composite measure as the dependent measure. There was a statistically significant effect of condition, $F(2, 87) = 6.24$, $p = .003$, $\eta_p^2 = .13$. Note that η_p^2 is the same as η^2 and η_G^2 for one-way ANOVA, and a value of .13 is a medium effect (Bakeman, 2005). Importantly, as predicted, children developed a better understanding of mathematical equivalence after receiving nontraditional practice (M = 1.17, SD = 1.09) than after receiving traditional practice and no extra practice (M = 0.47, SD = 0.81), $F(1, 87) = 11.76$, $p = .001$, Cohen's d = 0.73. A Cohen's d value of 0.73 is a medium-to-large effect (Cohen, 1992). There was no statistical difference in understanding of mathematical equivalence after receiving traditional practice (M = 0.37, SD = 0.72) versus no extra practice (M = 0.57, SD = 0.90), $F(1, 87) = 0.72$, $p = .40$.

Given that scores on the composite measure were not normally distributed, we also performed a nonparametric analysis to ensure that the observed effects did not depend on the method of analysis. We first classified children according to whether or not they scored at least one point on the composite measure of understanding of mathematical equivalence. Across conditions, 40% of children scored at least one point. We then used binomial logistic regression to predict the log of the odds of scoring at least one point on the measure (see Agresti, 1996). Two Helmert contrast codes were used to represent the three levels of condition (1) nontraditional practice versus the two "control" conditions and (2) traditional practice versus no extra practice. Results were consistent with the ANOVA. As predicted, participants in the nontraditional practice condition were more likely than participants in the other two conditions to score at least one point on the measure (18 of 30 [60%] versus 18 of 60 [30%]), $\hat{\beta} = 1.26$, $z = 2.69$, *Wald* $(1, N = 90) = 7.22$, $p = .007$. The model estimates that the odds of scoring at least one point on the measure are more than three and a half times higher after participating in the nontraditional practice condition than after participating in one of the other conditions. An odds ratio of 3.5 is a medium-to-large effect (Haddock, Rindskopf, & Shadish, 1998). There was no statistical difference between the traditional practice and no extra practice conditions (8 of 30 [27%] versus 10 of 30 [36%]), $\hat{\beta} = 0.32$, $z = 0.56$, *Wald* $(1, N = 60) = 0.32$, $p = .57$.

Results were similar at follow-up. Children's performance in the presence of brief tutelage and feedback at follow-up was relatively good compared with how children typically perform on mathematical equivalence problems ($M = 1.82$, $SD = 1.69$). Consistent with predictions, children who had participated in the nontraditional practice condition solved more mathematical equivalence problems correctly ($M = 2.33$, $SD = 1.73$) than did children who had participated in the traditional practice condition ($M = 1.30$, $SD = 1.53$), $F(1, 58) = 6.07$, $p = .02$, $\eta_p^2 = .10$.

Importantly, the gains in understanding of mathematical equivalence (shown above) did not appear to be accompanied by any detectable decrements in computational fluency. Children in the nontraditional practice condition had a similar average percentile rank ($M = 52.70$, $SD = 24.56$) to children in the traditional practice ($M = 52.93$, $SD = 29.65$) and no extra practice ($M = 51.20$, $SD = 27.80$) conditions on Level 8 of the Math Computation section of the Iowa Tests of Basic Skills, $F(2, 87) = 0.035$, $p = .96$. They also performed similarly when solving single-digit addition facts: accuracy, $F(2, 87) = 0.62$, $p = .54$, and average reaction time, $F(2, 87) = 0.18$, $p = .83$.

We performed a more targeted analysis of how the conditions affected computational fluency on the single-digit addition facts, and it produced similar findings. We compared children's performance on the three most difficult single-digit addition problems used in our assessment (9 + 8, 7 + 9, and 8 + 6) to their performance on three easier problems that were matched to the difficult problems based on one of the addends (9 + 2, 7 + 5, and 4 + 6). Note that the practice sessions were structured so that children received nearly twice as much practice with the difficult problems as they did with the matched problems. Table 2 displays the average reaction time and accuracy on each problem type by condition. Accuracy on both problem types was near ceiling in all conditions, so we focused our analysis on reaction time. We performed a 3 (condition: traditional practice, nontraditional practice, no extra practice) x 2 (problem type: difficult versus matched) mixed-factor analysis of variance (ANOVA) with repeated measures on problem type, and average reaction time as the dependent measure. Not surprisingly, average reaction time was slower on the difficult problems than it was on the matched problems, $F(1, 87) = 33.25$, $p < .001$, $\eta_p^2 = .28$. The main effect of condition was not statistically significant, $F(2, 87) = 0.37$, $p = .69$, but the anticipated interaction between problem type and condition was statistically significant, $F(2, 87) = 3.15$, $p = .048$, $\eta_p^2 = .07$. As shown in the table, children in the two practice conditions were faster than children in the no extra practice condition on the difficult problems, but children in all conditions had similar reaction times on the easier, matched problems. Children in the nontraditional condition had similar reaction times to children in the traditional condition on both problem types.

Discussion

The present study provides the first well-controlled evidence that children's understanding of mathematical equivalence can be improved simply by modifying the format in which arithmetic problems are presented during practice. Results suggest that children who practice problems presented in a nontraditional format (e.g., 17 = 9 + 8) develop a better understanding of mathematical equivalence than children who practice problems presented in the traditional format (e.g., 9 + 8 = 17). Importantly, these improvements in understanding do not seem to be accompanied by any detectable decrements in computational fluency.

Given the extent to which children are exposed to arithmetic in the early school years, it is not surprising that they become well versed in the patterns routinely encountered in arithmetic problems (McNeil & Alibali, 2005b). Indeed, many studies have shown that children possess powerful learning mechanisms that enable them to pick up on stable structure when it is present in a domain (e.g., Fiser & Aslin, 2002; Gentner & Medina, 1998; Gomez, 2002; Saffran, 2003; Sheya & Smith, 2006; Sloutsky & Fisher, 2008). By detecting and extracting the stable patterns in a domain, learners can construct long-term memory representations to serve as the default representations in that domain so unnecessary computations can be circumvented in the future (cf. Salthouse, 1991). Although these default representations can be helpful at times (e.g., Chase & Simon, 1973), they also can become entrenched, thus increasing learners' resistance to change (e.g., McNeil & Alibali, 2005b; Zevin & Seidenberg, 2002; Bruner, 1957; Luchins, 1942).

In the case of children's representations of arithmetic, there are at least three entrenched patterns: the "operations on left side" problem format, the "perform all given operations on all given numbers" strategy, and the "calculate the total" concept of the equal sign (McNeil & Alibali, 2005b). Although children's internalization of these operational patterns leads to fast and accurate performance on traditional arithmetic problems (e.g., 3 + 4 = __, McNeil & Alibali, 2004), it hinders understanding of and performance on mathematical equivalence problems, which overlap with, but do not map directly onto the patterns (McNeil & Alibali, 2004, 2005b).

Results of the present study suggest that meaningful practice with problems presented in a nontraditional format helps children improve their understanding of mathematical equivalence. We have argued that this is because nontraditional practice weakens the entrenchment of the overly narrow operational patterns and exposes children to patterns that facilitate (rather than hinder) acquisition of the to-be-learned concept. Every time children are meaningfully exposed to an arithmetic problem that has the operations to the right side of the equal sign, they increase the number of stored instances that do not correspond to the entrenched, “operations on left side” pattern. Eventually, children’s representation of the “operations on left side” format loses its predictive power, and children become less likely to activate it automatically every time they encounter a mathematics problem. Subsequently, when children are presented with a mathematical equivalence problem, they are less likely to rely on their representations of the overly narrow operational patterns and are more likely to encode the problem correctly and reflect on an appropriate solution strategy that takes into account the novel problem structure.

In the present study, we administered nontraditional practice to children ages 7-8 because their representations of the operational patterns have not yet been strengthened to their most entrenched levels (see McNeil, 2007). The observed benefits of nontraditional practice may be even greater for younger children (e.g., ages 5-6), who have less established representations of the operational patterns. Moreover, the benefits may be weaker for older children (e.g., ages 9-10), who have already strengthened representations of the operational patterns to their most entrenched levels. Once representations of the operational patterns are fully entrenched, nontraditional arithmetic practice may not be enough on its own to weaken them enough to enable children to develop a correct understanding of mathematical equivalence.

Along with weakening children's representations of the operational patterns, there are at least two additional processes by which meaningful practice with nontraditional problem formats could help children develop an understanding of mathematical equivalence. First, the sheer novelty of a nontraditional problem format may bolster children's attention during practice and lead them to be more *mindful* of what they are practicing. Langer (2000) defines mindfulness as "a flexible state of mind in which [one is] actively engaged in the present, noticing new things and sensitive to context" (p. 220). This open and adaptable perspective may reduce resistance to change and make it more likely for problem solvers to encode novel aspects of a problem, choose the most appropriate solution strategy, and extract relevant conceptual information. Indeed, in Luchins's (1942) water jar experiments, participants became less resistant to change their use of a familiar, suboptimal strategy and more likely to choose the most appropriate strategy after receiving a simple warning to be mindful. It is possible that the nontraditional problem format serves as a visual warning for children to be mindful, and in turn, helps them gain conceptual knowledge from their arithmetic practice.

Second, meaningful practice with nontraditional formats may enhance understanding of mathematical equivalence because it diverges from children's established knowledge and provides an opportunity for cognitive conflict. Indeed, according to Piaget (1980, cf. Inhelder, Sinclair, & Bovet, 1974; VanLehn, 1996), cognitive conflict is the primary impetus for cognitive change. When children see operations to the right side of the equal sign, it conflicts with their established knowledge of arithmetic. It is possible that this conflict forces children to adjust their thinking to accommodate the nontraditional format (but see Karmiloff-Smith, 1984; Siegler & Jenkins, 1989 for evidence that cognitive change occurs in the absence of cognitive conflict).

Overall, the present study adds to the evidence suggesting that early experience, rather than general conceptual or working memory limitations in childhood, is a primary factor behind children's difficulties with mathematical equivalence. We have long known that even young children can understand mathematical equivalence when given targeted, conceptual instruction (Carpenter et al., 2000; De Corte & Verschaffel, 1981; Jacobs et al., 2007; Saenz-Ludlow & Walgamuth, 1998), or when given special instruction designed to circumvent the working memory demands of mathematical equivalence problems (e.g., Case, 1978). However, the present study is the first to show that children can develop a better understanding of mathematical equivalence, even without explicit instruction. Moreover, it is the first to show that practice with arithmetic can be re-structured in a way that helps children develop conceptual understanding *at the same time as* they work to improve computational fluency.

More broadly, results support the idea that relatively small changes in the structure of the environment can affect the development of children's understanding of important concepts. Researchers have a history of showing that macro-level differences in children's environment, such as differences in socio-economic status or quality of the early childcare setting, have the potential to affect cognitive development (Bradley & Corwyn, 2002; Tran & Weinraub, 2006). More recently, however, we have started to recognize that even differences in relatively specific, micro-level factors, such as the amount of mathematically-relevant speech used in preschool classrooms or how often young children play number board games, can exert large effects on children's cognitive development (e.g., Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006; Ramani & Siegler, 2008). According to this view, even seemingly minor differences in the scope, sequence, or format of input may snowball to yield substantial differences in the development of children's understanding of important concepts, particularly in mathematics (cf.

Li et al., 2008).

Despite the positive effects of the nontraditional problem format observed in this study, at least four questions remain unanswered. First, why did the nontraditional condition *not* lead to greater gains in understanding of mathematical equivalence? On average, children in the nontraditional condition solved and reconstructed fewer than half of the mathematical equivalence problems correctly, and only 23% defined the equal sign relationally (see Table 1). Although their performance was better than that of children in the other two conditions, it was far from ideal. These findings suggest that the operational view is resistant to change (cf. Alibali, 1999; McNeil, 2008). One possibility is that children develop an operational view even before the start of formal schooling (Falkner et al., 1999) based on their informal interpretation of addition as a unidirectional process (Baroody & Ginsburg, 1983). According to this perspective, arithmetic problems may activate the operational view to some degree, regardless of format. If this is true, then we may not be able to eradicate the operational view simply by exposing children to different arithmetic problem formats (Denmark et al., 1976). Instead, it may be necessary to expose children to the equal sign outside of an arithmetic context (e.g., 28 = 28) first, so they can solidify a relational view before moving on to a variety of arithmetic problem formats (Baroody & Ginsburg, 1983; Denmark et al., 1976; McNeil, 2008; Renwick, 1932).

Second, how does extended practice with the nontraditional problem format affect children's computational fluency? Children who participated in our practice conditions received approximately 100 minutes of supplementary practice with arithmetic. Within this schedule, the traditional and nontraditional practice conditions produced similar levels of computational fluency, but it is possible that the two conditions would diverge after extended practice. Because our participants already had a few years of experience with traditional arithmetic practice, it is unclear if comparable levels of computational fluency would have resulted if children had been presented with both formats right from the start of schooling. Variability in practice can lead to wider generalization but slower learning (Brown, Kane & Echols, 1986; Chen, 1999); thus, exposing children to both formats from the start of schooling could slow their learning of basic arithmetic facts (but see Carpenter et al., 1989 for an alternative view). However, it is important to note that even if learning of basic facts was slowed to some degree, it would be a relatively small price to pay for wider generalization and an easier transition to upper-level mathematics.

Third, what are the long-term consequences of improving children's understanding of mathematical equivalence? One of the practical motivations behind research in this area is the need for better "preparation of students for entry into, and success in, Algebra" (National Mathematics Advisory Panel, 2008). We found that practice with the nontraditional format improved children's understanding of mathematical equivalence, and these improvements continued to be evident two weeks later. However, the longer-term consequences of such improvements have never been tested. Future studies should investigate the consequences of developing a better understanding of mathematical equivalence in the early grades and whether or not it translates to greater success in upper-level mathematics classes, including Algebra.

Finally, what are the effects of nontraditional practice in a classroom setting? The present study was a tightly controlled experiment in which children practiced arithmetic one-on-one with an experimenter who stuck to a meticulous script. More typical learning environments are often less structured and less conducive to one-on-one instruction. In order to determine the practical effectiveness of nontraditional arithmetic practice, we will need to investigate whether the results generalize to children practicing arithmetic in a classroom setting.

Conclusion

A primary goal of research in cognitive development is to understand the construction of knowledge and how it changes over time. The present study addressed this goal by providing evidence to suggest that the difficulties children have with particular concepts may not always be caused by something children lack relative to adults, such as general conceptual structures or working memory resources. Instead, difficulties can emerge as a consequence of prior learning in the target domain. Specifically, difficulties can emerge when aspects of the to-be-learned concept overlap with, but do not map directly onto, the patterns that children learn from their early experience in the domain. These findings suggest that parents and educators may want to pay attention to the structure of children's early learning environments to make sure children are not being consistently exposed to narrow patterns that constrain future learning.

Importantly, the present findings create a bridge between developmental theory and a growing literature in mathematics education that suggests that children should not be exposed to arithmetic facts presented in a single, traditional format as they are year after year in classrooms across the United States (Baroody & Ginsburg, 1983; Carpenter et al., 2003; McNeil et al., 2006; Seo & Ginsburg, 2003). We have shown here that a simple modification to the problem format may provide a more suitable learning context that can enhance the development of children's understanding of mathematical equivalence. This finding suggests that problem contexts may exert a larger effect on children's cognitive developmental processes than once thought, and that general conceptual or working memory limitations, though important, cannot fully explain children's misconceptions in mathematics. To better understand the construction of knowledge and how it changes over time, it is essential to continue investigating how children's early experiences in a domain shape and constrain the path of development.

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Author Note

This research was supported by the Institute of Education Sciences, U.S. Department of Education through Grant R305B070297 to McNeil. The opinions expressed are those of the authors and do not represent views of the Institute or the U.S. Department of Education. We thank Martha Alibali for feedback on the method and for comments on a previous version of this paper. Thanks also to Matt Gibson, Krysten Williams, Noelle Crooks, and Jenny Heil for help with coding and tutoring. This research would not have been possible without the support of many administrators, teachers, parents, and students in the greater South Bend area.

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Table 1.

Performance on each of the measures of understanding of mathematical equivalence by condition

Task & performance measure	Nontraditional	Traditional	No extra practice
Equation solving			
M (*SD*)	1.43 (1.72)	0.33 (1.06)	0.57 (1.14)
% above average	47	10	23
Equation encoding			
M (*SD*)	1.47 (1.41)	0.87 (1.22)	0.80 (0.96)
% above average	47	23	23
Defining the equal sign			
% who defined relationally	23	3	10

Table 2.

Accuracy and reaction time on the difficult and matched addition problems by condition.

Problem type & performance measure	Nontraditional	Traditional	No extra practice
Difficult problems			
M reaction time in seconds (*SD*)	7.64 (4.08)	6.98 (3.86)	9.16 (6.80)
% correct (*SD*)	92 (14)	90 (25)	86 (26)
Matched problems			
M reaction time (*SD*)	5.36 (2.86)	5.45 (3.40)	4.84 (2.71)
% correct (*SD*)	96 (12)	93 (16)	93 (16)

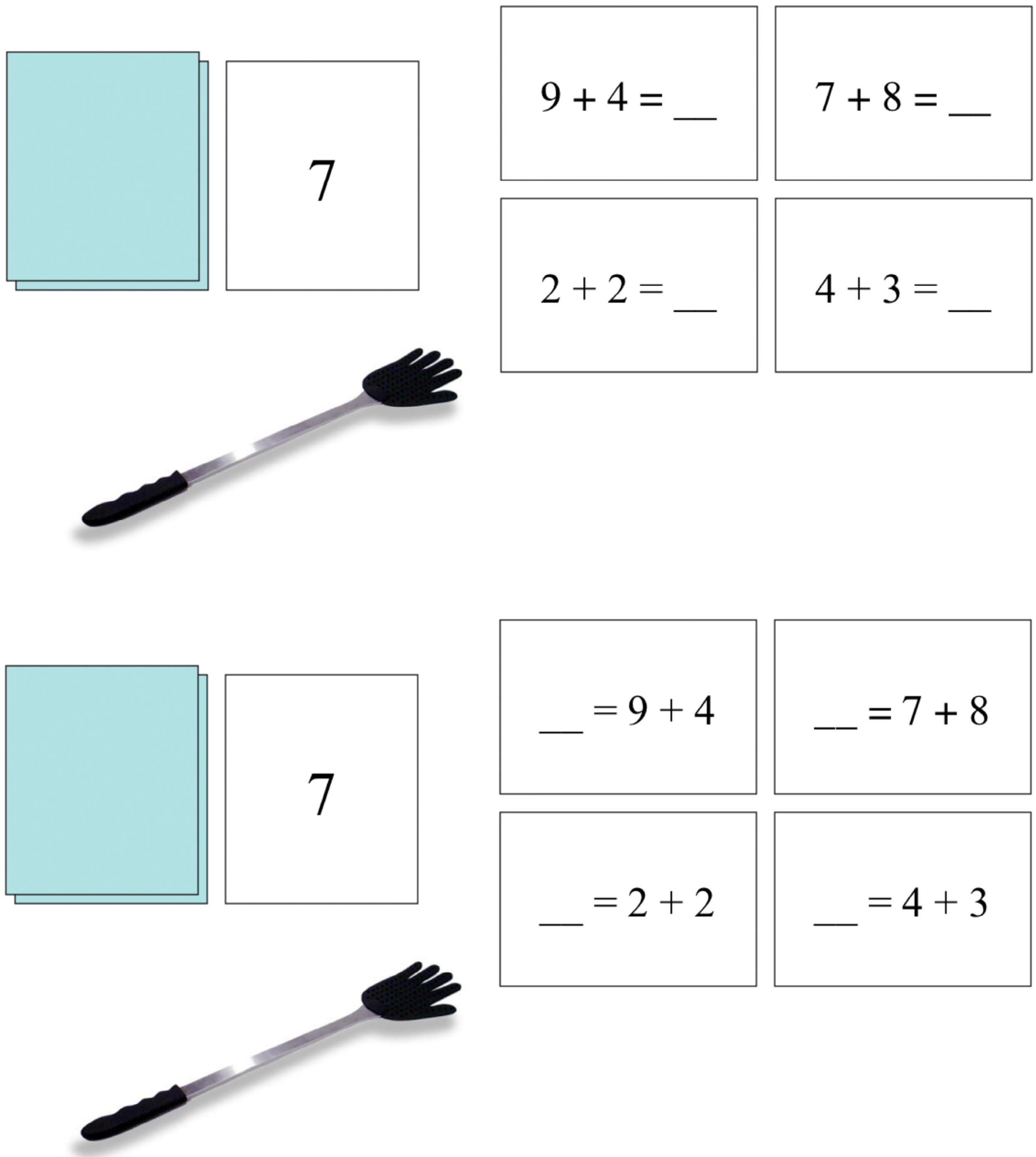

Figure 1. Illustration of the traditional (top panel) and non-traditional (bottom panel) versions of one of the games used in the practice sessions.